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LETTER TO THE EDITOR

Conformal invariance studies of the Baxter–Wu model and a related site-colouring problem

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Abstract. The partition function of the Baxter–Wu model is exactly related to the generating function of a site-colouring problem on a hexagonal lattice. We extend the original Bethe ansatz solution of these models in order to obtain the eigenspectra of their transfer matrices in finite geometries and general toroidal boundary conditions. The operator content of these models is studied by solving numerically the Bethe-ansatz equations and by exploring conformal invariance. Since the eigenspectra are calculated for large lattices, the corrections to finite-size scaling are also calculated.

1. Introduction

The Baxter–Wu model is defined on a triangular lattice by the Hamiltonian

$$H = -J \sum_{\langle ijk \rangle} \sigma_i \sigma_j \sigma_k \tag{1}$$

where the sum extends over the elementary triangles and $\sigma_i = \pm 1$ are Ising variables located at the sites. This model is self-dual [1] with the same critical temperature as that of the Ising model on a square lattice, and was solved exactly in its thermodynamic limit by Baxter and Wu [2]. Its leading exponents [2], $\alpha = 2/3$, $\mu = 2/3$ and $\eta = 1/4$, are the same as those of the four-state Potts model [3–5]. Due to this and the fact that both models have a fourfold degenerate ground state, it was conjectured that they share the same universality class of critical behaviour. However, from numerical studies of these models on a finite lattice it is well known that both models show different corrections to finite-size scaling. Whereas in the Potts models [6–9] these corrections are governed by a marginal operator, producing logarithmic corrections with the system size, this is not the case in the Baxter–Wu model [10, 11]. This raises the question of knowing which operator governs these corrections in the Baxter–Wu model.

With the developments of conformal invariance applied to critical phenomena [12], two models are considered in the same universality class of critical behaviour only if they have the same operator content, not only the leading critical exponents. The operator content of the four-state Potts model was already conjectured from finite-size studies in its Hamiltonian formulation [13], and can be obtained by a $Z(2)$ orbifold of the Gaussian model (see [14] for a review).

In this letter, by exploiting the conformal invariance at the critical point, we report on our calculation of the operator content of the Baxter–Wu model. In order to do this calculation we have to generalize the original Bethe ansatz solution of the model, since this

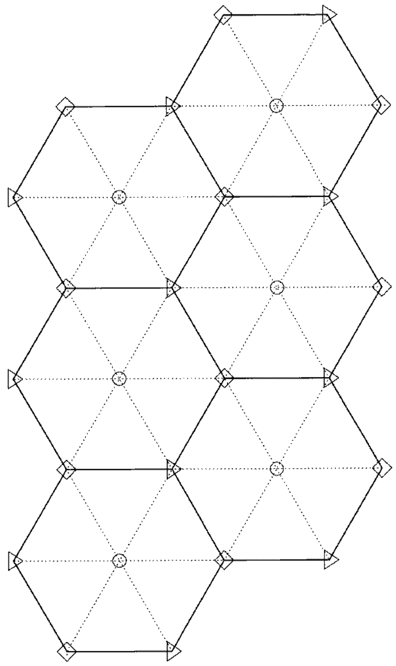

Figure 1. The Baxter–Wu model is defined on the triangular lattice formed by the points ○, ♢ and △. The site-colouring problem (SCP) is defined on the hexagonal lattice formed by the points △ and ♢, connected by continuous lines.

solution does not give the complete eigenspectrum of the associated transfer matrix T. The conformal anomaly c and anomalous dimensions $(x_1, x_2, \ldots)$ are obtained in a standard way from the finite-size behaviour of the eigenspectra of the associated transfer matrix, at the critical temperature. If we write $T = \exp(-\hat{H})$, then in a strip of width L with periodic boundary conditions the ground-state energy, $E_0(L)$, of $\hat{H}$ behaves for large L as [15]

$$\frac{E_0(L)}{L} = \epsilon_\infty - \frac{\pi c v_s}{6L^2} + o(L^{-2}) \tag{2}$$

where ϵ_∞ is the ground-state energy, per site, in the bulk limit. Moreover, for each operator O_α with dimension x_α there exists a tower of states in the spectrum of $\hat{H}$ with eigenenergies given by [12, 16]

$$E^{\alpha}_{m,m'}(L) = E_0 + \frac{2\pi v_s}{L}(x_\alpha + m + m') + o(L^{-1}) \tag{3}$$

where $m, m' = 0, 1, 2, \ldots$. The factor v_s appearing in (2) and (3) is the sound velocity and has unit value for isotropic square lattices. The higher eigenvalues of T can be calculated directly by numerical diagonalization. However, since T is not sparse and has dimension 2^L, for a horizontal width L, we cannot compute its eigenspectra by numerical diagonalization methods for $L > L_0 \sim 26$. Instead of a direct calculation we relate this problem to a *site-colouring problem* (SCP) on a hexagonal lattice, which can be solved by the Bethe ansatz. Following [2] the partition function, $Z^{\mathrm{BW}}_{L\times N}$, of the Baxter–Wu model on a periodic triangular lattice with $L(N)$ rows (columns) in the horizontal (vertical) direction is related to the partition function, or generating function, $Z^{\mathrm{SCP}}_{M\times N}$, of a SCP on a hexagonal lattice with

$M = 2L/3(N)$ rows (columns) in the horizontal (vertical) direction. In the limit $N \to \infty$,

$$Z^{\mathrm{BW}}_{L\times N} = Z^{\mathrm{SCP}}_{M\times N}. \tag{4}$$

In figure 1 the Baxter–Wu model is defined on the triangular lattice formed by connecting the points ○, ◊ and △. The related SCP is defined on the hexagonal lattice formed by the points △ and ◊, connected by continuous lines. The configurations in the SCP are defined by attaching at the points of the hexagonal lattice (i, j) $(i = 1, \ldots, M;\ j = 1, \ldots, N)$ site-colour variables $c_{i,j} = 1, 2, \ldots, 8$, satisfying the constraint that any two nearest-neighbour colours must differ by 1 or -1. The partition function Z^{SCP} given in (4) is obtained by adding all the colour configurations with weights given by the product of the fugacities $z_j (j = 1, 2, \ldots, 8)$ of each colour in the lattice configuration. These fugacities are given by

$$\begin{aligned} &z_1 = z_3 = z_5 = z_7 = 2\sinh(4\beta J) \\ &z_2^{-1} = z_4 = z_6^{-1} = z_8 = \sinh(2\beta J) \equiv t. \end{aligned} \tag{5}$$

The critical point of the Baxter–Wu model and of the SCP is given by the self-dual point $t = t_{\mathrm{c}} = 1$. If we write $T = \exp(-\hat{H})$ for both models and since $Z = \mathrm{Tr}(T^N)$, the relation (4) implies

$$\mathrm{Tr}(\mathrm{e}^{-NH_L^{\mathrm{BW}}}) = \mathrm{Tr}(\mathrm{e}^{-NH_M^{\mathrm{SCP}}}). \tag{6}$$

It is important to observe that although H_L^{BW} and H_M^{SCP} have the same dimension 2^L they may have different eigenvalues.

In the SCP we say we have a dislocation [2] in a given link wherever the colour in its right end is smaller than the left one. The number n of dislocations in a given row is a conserved quantity. Consequently, the Hilbert space associated to T^{SCP} or H^{SCP} is separated into block disjoint sectors labelled by the values of n. For a periodic hexagonal lattice of width M (even) the possible values of n are even and given by $M, M \pm 4, M \pm 8, \ldots, M \pm 4\,\mathrm{int}(M/4)$. Due to (6) the SCP also has an additional $Z(2)$ symmetry (eigenvalues $\epsilon = \pm 1$), since adding 4 (modulo 8) to all colours in a given configuration does not change its weight in the partition function. The Bethe ansatz solution presented by Baxter and Wu [2] only gives part of the eigenspectrum of H^{SCP}, since they considered only eigenstates which are even under this symmetry. We generalized [17] their solution in order to obtain the missing odd eigenvectors. For the sake of brevity we only present here the Bethe ansatz equations. The eigenvalues $E_n^{(s_j)}$ of H^{SCP} in the sector with n dislocations are given by

$$E_n^{\{s_i\}} = -\frac{M}{4}\ln(16t^2(1+t^2)) - \sum_{j=1}^{n}(e_j^{(s_j)} - ik_j^{(s_j)}) \tag{7}$$

where

$$e_j^{(s_j)} = 1/2\ln(x_j + s_j\sqrt{x_j^2 - 1}) \qquad x_j = \cos(2k_j^{(s_j)}) + t + 1/t \tag{8}$$

with $1 = s_1 = s_2 = \cdots = s_{n-l} = -s_{n-l+1} = \cdots = -s_n$, and $l = 0, 1, \ldots, n$ fixed. The quasimomenta $\{k_j^{(s_j)}\}$ are obtained by solving the equations

$$\exp(\mathrm{i}Mk_j^{(s_j)}) = -\sqrt{\epsilon}\prod_{p=1}^{n}\left(\frac{\cosh(e_j^{(s_j)} + \mathrm{i}k_p^{(s_p)})}{\cosh(e_p^{(s_p)} + \mathrm{i}k_j^{(s_j)})}\right) \qquad j = 1, 2, \ldots, n \tag{9}$$

where $\epsilon = \pm 1$. The value $\epsilon = 1$ gives the part of the eigenspectrum which is even under the $Z(2)$ symmetry and was derived in [2], while $\epsilon = -1$ gives the odd part of the eigenspectrum. Strictly speaking this is a conjecture since the completeness of the Bethe ansatz solutions is always a difficult question.

Numerically we have studied these equations extensively at the critical point $t = t_c = 1$, for general values of n, ϵ and l and for lattice sizes up to $M \sim 200$. For example, in the eigensector where $n = M$, by setting $l = 0$ we obtain the energies of the ground state and first excited state by choosing $\epsilon = 1$ and $\epsilon = -1$, respectively.

Table 1. Conformal anomaly estimators c_M, as a function of M, for the SCP and Baxter–Wu model.

M	c_M
6	0.996 590 995
10	0.998 910 268
50	0.999 959 561
100	0.999 989 915
150	0.999 995 519
200	0.999 997 480

Let us calculate the conformal anomaly by using (2). The bulk energy $\epsilon_\infty^{\rm SCP} = -\frac{3}{4}\ln 6$ can be obtained from the solution in the bulk limit [2] and the sound velocity $v_s = \sqrt{3}/3$, can be inferred from (3) and an overall analysis of the dimensions appearing in the model. Using these values in (2) we obtain the finite-size sequence estimators $c(M)$ for the conformal anomaly, presented in table 1. As expected the conformal anomaly is $c = 1$, as for the four-state Potts model. The relation (6) does not imply that in a finite lattice the eigenergies of $H_L^{\rm BW}$ and $H_M^{\rm SCP}$ are the same. In fact this is the case. Fortunately by comparing their eigenspectra by direct calculations on small lattices we verify that many of the lower energies, including the ground-state energy, are exactly the same. Consequently by using the bulk limit value [2] $\epsilon_\infty^{\rm BW} = -\frac{1}{2}\ln 6$ we obtain the same sequence shown in table 1. The sound velocity which comes from our analysis is now $v_s = \sqrt{3}/2$ and the conformal anomaly has the expected value $c = 1$.

Table 2. Scaling dimensions estimators $x_j^\epsilon(M-n, l)$, as a function of the lattice size M, for some eigenenergies. These energies are the jth lowest energy obtained by solving (7)–(9) with values n, ϵ and l.

M	$x_1^-(0,0)$	$x_2^-(1,0)$	$x_3^+(0,1)$	$x_2^+(2,0)$	$x_4^+(0,1)$
6	0.125 028 03	0.248 967 41	0.506 262 26	0.626 135 04	0.986 483 57
10	0.125 017 02	0.249 597 71	0.502 153 17	0.625 483 22	0.996 989 67
50	0.125 000 83	0.249 983 23	0.500 084 06	0.625 020 93	0.999 921 01
100	0.125 000 21	0.249 995 80	0.500 020 99	0.625 005 24	0.999 980 57
150	0.125 000 09	0.249 998 13	0.500 009 33	0.625 002 33	0.999 991 39
200	0.125 000 05	0.249 998 95	0.500 005 25	0.625 001 31	0.999 995 16

The large-L behaviour of the energies of excited states will give the operator content of the models. Using (3), the finite-size sequences obtained for some dimensions associated to $H^{\rm SCP}$ are shown in table 2. The estimators $x_j^\epsilon(M-n, l)$ are the jth lowest eigenenergy obtained by solving (7)–(9) with the values n, ϵ and l. As a result of an extensive calculation of the eigenspectra of $H^{\rm SCP}$, obtained by direct diagonalization on small lattices and by solving (7)–(9) for large lattices, we arrive at the following conjecture. Namely the dimensions of primary operators in a given sector labelled by $n = M + 4p$ are given by

$$x_{p,q} = \tfrac{1}{2}(4p^2 + \tfrac{1}{4}q^2) \qquad q = 0, \pm 1, \pm 2, \ldots \tag{10}$$

where $p = 0, \pm 1, \pm 2, \ldots$ for the periodic lattice. The number of descendants, with dimensions $x_{p,q} + m + m'$ $(m, m' \in \mathcal{Z})$ is given by the product of two independent Kac–Moody characters. The dimensions (10) are similar, in a Gaussian model [18] with compactification radius equal to 2, to the dimensions of an operator with vorticity p and spin-wave number q. The Gaussian model at this radius corresponds to the continuum limit Kosterlitz–Thouless point of the X–Y model in the torus [14, 19]. This implies that the SCP belongs to the same universality class as the Kosterlitz–Thouless phase transition of the X–Y model.

We have also studied the SCP with more general toroidal boundary conditions which preserves the same symmetries as the periodic case (n and ϵ are good quantum numbers). Those boundary conditions are obtained by imposing to each row ($j = 1, 2, \ldots$) of a colour configuration in the SCP the constraint $c_{M+1,j} = c_{1,j} + k$, where $k = 0, 2, 4$ or 6. The periodic case is obtained when $k = 0$. The possible values of n are now given by $n = M + 4p$ where $p = j - \frac{1}{8}k$ $(j = 0, \pm 1, \pm 2, \ldots)$. In this case $k = 2$ and $k = 6$ we were not able to apply the Bethe ansatz for arbitrary temperatures, but only at the critical temperature $t = t_c = 1$ [17]. The Bethe ansatz equations turn out to be the same as in (7)–(9) but now the values of n depend on the boundary condition. If in (2) we take $E_0(M)$ as the ground-state energy of the periodic lattice ($k = 0$) our numerical solutions of equations (7)–(9) also give the dimensions (10) but with $p = j - \frac{1}{8}k$ $(j = 0, \pm 1, \pm 2, \ldots)$.

Let us return to the Baxter–Wu model. In this case by comparing the eigenspectra of H^{BW} and H^{SCP}, obtained by a direct diagonalization on small lattices, we verify that many of the dimensions $x_{p,q}$ appearing in (10) are absent. For example, the energies producing the estimators in the second and eighth columns of table 2 only appear in H^{SCP}. Following, for large lattices, the energies which are exactly related in both models, we verified that the lower dimensions in the Baxter–Wu model are given by $x = 0, \frac{1}{8}, \frac{1}{2}, \frac{9}{8}, \ldots,$ and appear with degeneracy $d_x = 1, 3, 1, 9, \ldots,$ respectively. These results, supplemented with the global eigenspectrum calculated for small systems, indicate that the operator content of the Baxter–Wu model is the same as that of the four-state Potts model [13] and is given in terms of a $Z(2)$ orbifold [14] of the Gaussian model. It is interesting to note that, whereas in the SCP the operator content is given in terms of characters of the Kac–Moody, algebra in the Baxter–Wu model the characters are those of the Virasoro algebra. Since the exact integrable SCP and the Baxter–Wu model belong to different universality classes some care has to be taken when we import exact results from the SCP to the Baxter–Wu model.

We also studied the Baxter–Wu model with more general toroidal boundary conditions which preserve its $Z(2) \otimes Z(2)$ symmetry. We observe numerically that the eigenenergies which appear in this case can also be obtained from the eigenspectrum of the SCP with the toroidal boundary conditions we considered ($k = 0, 2, 4, 6$). Calculating the corresponding dimensions for large lattices we obtain the same dimensions reported in [13] for the four-state Potts model. These results imply that both models are indeed in the same universality class, being governed at the critical point and arbitrary toroidal geometry by the same conformal field theory.

Since we calculate eigenenergies of H^{BW} and H^{SCP} for large lattices we can now also calculate the corrections to finite-size scaling for both models. Consider the lowest eigenenergy E_α, associated to an operator with dimension x_α. From (3) the correction

Table 3. Estimators of the exponent x_γ in (12) of the dominant correction of some eigenenergies E_α with dimension x_α.

M	$x_\alpha = 0$	$x_\alpha = 0.125$	$x_\alpha = 0.25$	$x_\alpha = 0.5$	$x_\alpha = 0.625$
20	4.083 6238	4.767 2008	3.953 517	4.027 4781	3.906 3776
50	4.046 6061	4.874 3240	3.990 329	4.057 6819	3.981 6761
100	4.003 5431	4.991 7793	3.998 189	4.001 0866	3.996 5769
150	4.001 2060	4.997 7042	3.999 406	4.000 3372	3.998 9165
200	4.000 5332	4.999 1458	3.999 680	4.000 1660	3.999 4630

$R_\alpha(M)$ associated to this level is given by

$$E_\alpha = Me_\infty + \frac{2\pi v_\mathrm{s}}{M}\left(x_\alpha - \frac{c}{12} + R_\alpha(M)\right). \tag{11}$$

According to conformal invariance [8, 12] R_α should behave as

$$R_\alpha(M) = \sum_\gamma \frac{a_\gamma}{M^{x_\gamma - 2}} + \sum_{\gamma,\gamma'} \frac{a_{\gamma\gamma'}}{M^{x_\gamma + x_{\gamma'} - 4}} \tag{12}$$

where $\{x_\gamma\}$ are the non-relevant dimensions ($x_\gamma \geqslant 2$) associated to the operators governing the finite-size corrections. In the four-state Potts model the lowest dimension x_γ in this set is associated to a marginal operator ($x_\gamma = 2$), and the corrections have a logarithmic behaviour with the system size. In table 3 we show our estimators for the dimension x_γ of the dominant correction for some eigenergies E_α, with corresponding dimension x_α, in the SCP and the Baxter–Wu model. In all these cases we clearly see that $x_\gamma = 4$, indicating that the corrections are integer powers. Rather than the four-state Potts model these corrections are like those of the Ising model. This explains why the finite-size studies of the Baxter–Wu model have good convergence, in contrast with the four-state Potts model.

These results show that although the four-state Potts model and the Baxter–Wu model share the same universality class of critical behaviour, having the same operator content, the finite-size effects correspond to different perturbations of the fixed point of the renormalization group. The SCP belongs to another universality class. This implies that not all exact results derived from the SCP should be translated to the Baxter–Wu model.

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